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THE ASYMPTOTIC FORM OF THE FAR FIELD OF AN INTERNAL WAVE SOURCE MOVING IN AN EXPONENTIALLY STRATIFIED MEDIUM†

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The uniform asymptotic form of the far field of a linear gravitational internal wave source which moves uniformly and rectilinearly along the horizontal in an homogeneous, horizontally exponentially stratified medium is constructed. The expression obtained enable one to find this asymptotic form for any mutual arrangement of the source and the observation point. Copyright © 1996 Elsevier Science Ltd.

1. FORMULATION OF THE PROBLEM

A space x, y, z is considered which is filled with an exponentially stratified, ideal, incompressible fluid with a density distribution $\rho(z) = \rho_0 \exp(-\sigma z)$ and the field, excited by a dipole mass source, which moves in a negative direction along the x axis at a constant velocity V. It is assumed that the source is switched on and commences its motion at $t = 0$. Then, when $t \to \infty$ and in the case of fixed $\xi = + Vt, y, z$ (that is, for a fixed position of the observation point relative to the source), the field tends to a finite limit. If the dipole is orientated along the x axis and has unit moment, the limiting values of the elevation ζ and the horizontal components of the velocity u_x, u_y have the form [1]

$$\zeta = V\frac{\partial^2 G}{\partial\xi\partial z}, \quad u_x = V^2\left(\frac{\partial^2}{\partial\xi^2} + k^2\right)\frac{\partial G}{\partial\xi}, \quad u_y = V^2\left(\frac{\partial^2}{\partial\xi^2} + k^2\right)\frac{\partial G}{\partial y} \tag{1.1}$$

$$G = \frac{-1}{8\pi^2 V^2}\iint\limits_{-\infty}^{\infty}\frac{\exp i(\alpha\xi + \beta y + \gamma|z|)d\alpha d\beta}{\sqrt{(\alpha^2+\beta^2)(k^2-\alpha^2)}}, \quad \gamma = \frac{1}{\alpha}\sqrt{(\alpha^2+\beta^2)(k^2-\alpha^2)} \tag{1.2}$$

where $k = \sqrt{(\sigma g)}/V$ and g is the acceleration due to gravity. The arithmetic value of the root is understood to be $\sqrt{(k^2 - \alpha^2)}$ when $k^2 > \alpha^2$ and $i\sqrt{(\alpha^2 - k^2)}$ when $k^2 < \alpha^2$.

The problem of determining the asymptotic form of the field in the far zone, that is, when $r = \sqrt{(\xi^2 + y^2 + z^2)} \gg 1$, is formulated. This asymptotic form has been obtained in [1] for the case when the phase function $\Phi = \alpha\xi + \beta y + \gamma z$ has stationary points, that is, when $\xi > 0$. The results in [1], however, are inapplicable when $\xi \to 0$, that is, in the neighbourhood of a plane which passes through the source and is perpendicular to its trajectory (a transverse plane) and when $z \to 0$, that is, in the neighbourhood of the horizon of the source. The purpose of this paper is to construct the asymptotic form of the far field which is applicable to any $kr \gg 1$ and ξ, y, z.

2. THE ASYMPTOTIC FORM OF THE FAR FIELD SUBJECT TO $|kz|$ HAVING A LOWER BOUND

We transform the integral for G. This integral can be written in the form

$$G = \frac{-1}{4\pi^2 V^2}\operatorname{Re}\int\limits_0^{\infty} d\alpha \int\limits_{-\infty}^{\infty} T(\alpha, \beta)d\beta$$

with the same integrand as in (1.2). Putting $\xi = r\cos\theta, y = r\sin\theta\cos\varphi, z = r\sin\theta\sin\varphi$ $(0 < \theta, \varphi < \pi)$ and changing to the variables of integration $p = d/k$; $q = \beta/\alpha$, we obtain

$$G = \operatorname{Re} Q; \quad Q = \frac{-1}{4\pi^2 V^2}\int\limits_0^{\infty}\frac{dp}{\sqrt{1-p^2}}\int\limits_{-\infty}^{\infty}\exp(ikr\Phi)\frac{dq}{\sqrt{1+q^2}} \tag{2.1}$$

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We now find the non-uniform asymptotic form Q as $kr \to \infty$ and for fixed $\theta \neq 0, \pi/2; \pi$; $\varphi \neq 0, \pi$. This asymptotic form is determined by the stationary point $T_1 = (\sin\varphi\cos\theta, -\cos\varphi \operatorname{ctg}\theta)$ of the phase function Φ within the domain of integration and the stationary point $T_2 = (0, 0)$ on the boundary of this domain and has the form

$$Q \approx -\frac{\exp(ikr\sin\varphi)\chi(\cos\theta)}{2\pi V^2 kr\sqrt{1-\sin^2\varphi\cos^2\theta}} + \frac{\exp i(\pi/4 + kr\sin\varphi\sin\theta)}{V^2(2\pi kr)^{3/2}\sqrt{\sin\varphi\sin\theta}\cos\theta} \tag{2.2}$$

The first term is the contribution of the stationary point T_1 and the second is the contribution of the stationary point T_2. The function $\chi(\cos\theta) = 1$ when $\cos\theta > 0$, that is, when this point is within the domain of integration $p > 0$ and makes a contribution to the asymptotics form Q; $\chi = 0$ when $\cos\theta < 0$, that is, when this point is outside the domain of integration. Hence, in the rear half-space $\xi > 0$ (with respect to the direction of the motion of the source), the field decreases when $r \to \infty$ as r^{-1}, while in the front half-space it decreases more rapidly as $r^{-3/2}$.

The asymptotic form (2.2) is inapplicable when $\cos\theta \to 0$, that is, for small $\xi = x + Vt$ in the neighbourhood of the transverse plane and when $\sin\varphi\sin\theta \to 0$, that is, for small $|kz| = kr\sin\varphi\sin\theta$ in the neighbourhood of the horizon of the source. We now write out the asymptotic form which is applicable close to the transverse plane, that is, which describes the transition from the front half-space to the rear half-space.

When $\cos\theta \to 0$, the stationary point T_1 tends to the boundary $p = 0$ of the integration domain. Since T_1 is the point of a local maximum in the phase function, the uniform asymptotic form G is expressed [2] in terms of the complex conjugate Fresnel integral. In order to obtain this integral, it is necessary to replace the function $\chi(\cos\theta)$ by the expression

$$F^*\left(\sqrt{2kr\sin\varphi}\sin(\pi/4-\theta/2)\right) + \frac{\exp i(\pi/4 - 2kr\sin\varphi\sin^2(\pi/4-\theta/2))}{2\sqrt{2\pi kr\sin\varphi}\sin(\pi/4-\theta/2)}$$

where $F^*(\eta)$ is the complex conjugate Fresnel integral

$$F^*(\eta) = \frac{\exp(\pi i/4)}{\sqrt{\pi}} \int_{-\infty}^{\eta} \exp(-is^2)\,ds$$

As a result, we obtain

$$G = \operatorname{Re} Q = \operatorname{Re}\left[-\frac{\exp(ikr\sin\varphi)F^*\left(\sqrt{2kr\sin\varphi}\sin(\pi/4-\theta/2)\right)}{2\pi V^2 kr\sqrt{1-\sin^2\varphi\cos^2\theta}\cos\theta} + \right.$$

$$\left. + \frac{\exp(\pi i/4 + ikr\sin\varphi\sin\theta)\left[\sqrt{1-\sin^2\varphi\cos^2\theta} - \sin(\pi/4+\theta/2)\sqrt{\sin\theta}\right]}{V^2(2\pi kr)^{3/2}\sqrt{\sin\varphi\sin\theta(1-\sin^2\varphi\cos^2\theta)}\cos\theta}\right] \tag{2.3}$$

This expression is applicable for values of θ close to $\pi/2$ and, when the argument of the Fresnel integral is large, it is asymptotically equivalent to (2.2). It is inapplicable, however, when $\sin\theta\sin\varphi \to 0$, that is, close to the horizon of the source.

3. THE ASYMPTOTIC FORM OF THE FAR FIELD IN THE FRONT HALF-SPACE FOR SMALL $|kz|$

Before constructing this asymptotic form, we transform the integral with respect to q in (2.1). We put ($q = \operatorname{sh} t$)

$$F(\alpha,\beta) = \int_{-\infty}^{\infty} \exp i\left(\alpha q + \beta\sqrt{1+q^2}\right)\frac{dq}{\sqrt{1+q^2}} = \int_{-\infty}^{\infty} \exp i(\alpha \operatorname{sh} t + \beta \operatorname{ch} t)\,dt$$

It can be shown by a shift with respect to t that, when $\alpha^2 - \beta^2 = \gamma^2 - \delta^2$, the equality $F(\alpha, \beta) + F(\gamma, \delta)$ holds. The integral with respect to q in (2.1) can therefore be written in the following equivalent forms

$$\int_{-\infty}^{\infty} \exp\left[ikr\sin\theta\left(pq\cos\varphi + \sqrt{(1-p^2)(1+q^2)}\sin\varphi\right)\right]\frac{dq}{\sqrt{1+q^2}} =$$

$$= \int_{-\infty}^{\infty} \exp\left[ikr\sin\theta\left(pq + \sqrt{1+q^2}\sin\varphi\right)\right]\frac{dq}{\sqrt{1+q^2}} =$$

$$= \int_{-\infty}^{\infty} \exp\left[ikr\sin\theta\left(q\cos\varphi + \sqrt{(1-p^2)(1+q^2)}\right)\right]\frac{dq}{\sqrt{1+q^2}} \tag{3.1}$$

Using the first of these equalities, we write

$$Q = -\frac{1}{4\pi^2 V^2}\int_0^{\infty}\frac{dp}{\sqrt{1-p^2}}\int_{-\infty}^{\infty}\exp ikr\left[p(\cos\theta + q\sin\theta) + \sqrt{1+q^2}\sin\theta\sin\varphi\right]\frac{dq}{\sqrt{1+q^2}} \tag{3.2}$$

Here, by analogy with formula (1.2), when $p > 1$, $\sqrt{(1-p^2)}$ is understood as being the quantity $i\sqrt{(p^2-1)}$, that is, the branch point $p = 1$ of the function $\sqrt{(1-p^2)}$ is circumvented in the lower half-plane during the integration.

We now transform the integration path in (3.2) into the half-line $p = t\exp(-i\beta)$ (where $0 < t < \infty$) in the complex p plane and the line $-\infty < q < \infty$ into a contour, consisting of the half-lines $q = i + t$ when $-\infty < t < 0$ and $q = i + t\exp(i\alpha)$ when $0 < t < \infty$, where $\alpha > \beta$. With this choice of integration contours and, when $\cos\theta \leqslant 0$, the exponential function in (3.2) will have a negative real part, the integral will be absolutely convergent when $|p|$, $|q| \to \infty$ and the order of integration in (3.2) can be changed. When $kr \gg 1$, the inner integral (with respect to p) is then calculated asymptotically and, for $G = \operatorname{Re} Q$, we obtain

$$G = \frac{-1}{4\pi^2 V^2 kr}\operatorname{Im}\int_{-\infty}^{\infty}\frac{\exp\left(ikr\sqrt{1+q^2}\sin\theta\sin\varphi\right)dq}{\sqrt{1+q^2}(\cos\theta + q\sin\theta)} + O(kr)^{-3}$$

where the pole $q = -\operatorname{ctg}\theta$ is circumvented in the upper half-plane. This function can be written in the form

$$G \approx \frac{-1}{8\pi^2 V^2 kr}\operatorname{Im}\int_{-\infty}^{\infty}\frac{\exp\left(ikz\sqrt{1+q^2}\right)}{\sqrt{1+q^2}}\left(\frac{1}{\cos\theta + q\sin\theta} + \frac{1}{\cos\theta - q\sin\theta}\right)dq =$$

$$= \frac{-\cos\theta}{4\pi^2 V^2 kr}\operatorname{Im}\int_{-\infty}^{\infty}\frac{\exp\left(ikz\sqrt{1+q^2}\right)dq}{\sqrt{1+q^2}(\cos^2\theta - q^2\sin^2\theta)} \tag{3.3}$$

where the pole $q = -\operatorname{ctg}\theta$ is circumvented in the upper half-plane and the pole $q = \operatorname{ctg}\theta$ is circumvented in the lower half-plane.

We will now prove that

$$\cos\theta\operatorname{Im}\int_{-\infty}^{\infty}\frac{\exp\left(i\zeta\sqrt{1+q^2}\right)dq}{\sqrt{1+q^2}(\cos^2\theta - q^2\sin^2\theta)} = -2\pi\left[\frac{J_0(\zeta)}{2} + \sum_{n=1}^{\infty}\left(-\operatorname{ctg}^2\left(\frac{\theta}{2}\right)\right)^n J_{2n}(\zeta)\right] \tag{3.4}$$

where J_{2n} is a Bessel function. Denoting the left-hand side of (3.4) is $F(\zeta, \theta)$, we have

$$F + \sin^2\theta\frac{\partial^2 F}{\partial\zeta^2} = \cos\theta\operatorname{Im}\int_{-\infty}^{\infty}\frac{\exp\left(i\zeta\sqrt{1+q^2}\right)dq}{\sqrt{1+q^2}} = \pi\cos\theta J_0(\zeta)$$

Hence, $F(\zeta, \theta)$ can be found as the solution of the equation

$$F + \sin^2\theta\frac{\partial^2 F}{\partial\zeta^2} = \pi\cos\theta J_0(\zeta)$$

which tends to zero as $|\zeta| \to \infty$ (which follows from the rule for circumventing the poles in integral (3.3) which has been formulated above). Formula (3.4) is now verified by direct computation using well-known formulae (see [3, formulae 8.471]) for the derivatives of Bessel functions.

Hence, when $kr \gg 1$ and $\cos\theta \leqslant 0$, we have obtained the following asymptotic expression for G

$$G = \frac{1}{2\pi V^2 kr}\left[\frac{J_0(kz)}{2} + \sum_{n=1}^{\infty}\left(-\operatorname{ctg}^2\frac{\theta}{2}\right)^n J_{2n}(kz)\right] + O(kr)^{-3} \tag{3.5}$$

This expression is convenient to use for small kz, that is, $J_{2n}(kz)$ tend to zero exponentially with respect to n when $n > kz/2$ and it suffices to take a relatively small number of terms in the series with respect to n. For example, when $kz = 4$, we have

$$J_0(4) = -0.3971 \quad J_4(4) = 0.2811 \quad J_8(4) = 0.0049$$
$$J_2(4) = 0.3641 \quad J_6(4) = 0.0490 \quad J_{10}(4) = 0.0002$$

and it is sufficient to take 3–4 terms in series (3.5). The multiplier $\mathrm{ctg}^{2n}(\theta/2)$ is an additional factor in the convergence, since $\mathrm{ctg}(\theta/2) < 1$ when $\pi/2 < \theta < \pi$. If $|kz| > 4$, the asymptotic expression (2.2) provides sufficient accuracy when the values of $|\cos\theta|$ have a lower bound while (2.3) provides sufficient accurately for small $|\cos\theta|$.

4. THE ASYMPTOTIC FORM OF THE FAR FIELD IN THE REAR HALF-SPACE

We now consider the values $\cos\theta > 0$. Integral (2.1), after making the substitution $p, q \to -p, -q$, can be written in the form

$$G = \mathrm{Re}\,Q; \quad Q = Q(r,\theta,\varphi) = \frac{-1}{4\pi^2V^2}\int_{-\infty}^{0}\frac{dp}{\sqrt{1-p^2}}\int_{-\infty}^{\infty}\exp(ikr\Phi_1)\frac{dq}{\sqrt{1+q^2}}$$

where

$$\Phi_1 = -p\cos\theta + pq\sin\theta\cos\varphi + \sqrt{(1-p^2)(1+q^2)}\sin\theta\sin\varphi$$

On the other hand, it is obvious that

$$Q(r,\pi-\theta,\varphi) = \frac{-1}{4\pi^2V^2}\int_{0}^{\infty}\frac{dp}{\sqrt{1-p^2}}\int_{-\infty}^{\infty}\exp(ikr\Phi_1)\frac{dq}{\sqrt{1+q^2}}$$

with the same function Φ_1. Hence

$$Q(r,\theta,\varphi) + Q(r,\pi-\theta,\varphi) = I(r,\theta,\varphi) = -\frac{1}{4\pi^2V^2}\int_{-\infty}^{\infty}\frac{dp}{\sqrt{1-p^2}}\int_{-\infty}^{\infty}\exp(ikr\Phi_1)\frac{dq}{\sqrt{1+q^2}}$$

and

$$G(r,\theta,\varphi) = -G(r,\pi-\theta,\varphi) + \mathrm{Re}\,I(r,\theta,\varphi)$$

Since the asymptotic form G when $\theta > \pi/2$ is already known, it is sufficient to find the asymptotic form I. We make use of the second equality in (3.1) after which we change the order of integration. The asymptotic form of the integral with respect to p can then be found by the stationary-phase method, after which we obtain

$$I \approx -\frac{\exp(-\pi i/4)}{(2\pi)^{3/2}V^2\sqrt{kr}}\int_{-\infty}^{\infty}\frac{\exp\left[ikr\left(q\sin\theta\cos\varphi + \sqrt{1+q^2\sin^2\theta}\right)\right]dq}{(1+q^2\sin^2\theta)^{1/4}\sqrt{1+q^2}} =$$

$$= I_1 = \frac{-\exp(-\pi i/4)}{(2\pi)^{3/2}V^2\sqrt{kr}}\int_{-\infty}^{\infty}\frac{\exp\left[ikr\left(q\cos\varphi + \sqrt{1+q^2}\right)\right]dq}{(1+q^2)^{1/4}\sqrt{\sin^2\theta+q^2}}$$

In the case of $|\sin\theta|$, which has a lower limit, that is, far away from the half-plane $\xi > 0$, the asymptotic form of the integral I_1 is calculated using the stationary-phase method

$$I \approx -\frac{\exp(ikr\sin\varphi)}{2\pi V^2 kr\sqrt{1-\sin^2\varphi\cos^2\theta}};$$

$$G(r,\theta,\varphi) \approx G_1(r,\theta,\varphi) = -\frac{\cos(kr\sin\varphi)}{2\pi V^2 kr\sqrt{1-\sin^2\varphi\cos^2\theta}} - G_2(r,\theta,\varphi) \tag{4.1}$$

$$G_2 = \frac{1}{2\pi V^2 kr}\left[\frac{J_0(kz)}{2} + \sum_{n=1}^{\infty}\left(-\mathrm{tg}^2\frac{\theta}{2}\right)^n J_{2n}(kz)\right] + O(kr)^{-3}$$

When $|\sin\theta| \to 0$ and the values of $|\cos\varphi|$ in the integrand in I_1 have a lower bound, the two branch points $q = \pm i\sin\theta$ tend to the real axis and it is necessary to take account of their contribution to the asymptotic form I.

In the principal term of the asymptotic form, this contribution is described by the model integral

$$\int_{-\infty}^{\infty} \frac{\exp(ikrq\cos\varphi)dq}{\sqrt{\sin^2\theta+q^2}} = 2K_0(kr\sin\theta\cos\varphi) = 2K_0(ky)$$

where K_0 is a Bessel function with an imaginary argument (see [3, formula 3.754]). Hence

$$G(r,\theta,\varphi) \approx G_1 - \frac{\cos(kr-\pi/4)K_0(ky)}{\pi^{3/2}V^2\sqrt{2kr}} \tag{4.2}$$

The second term is only important when $|ky|$ is bounded. When $|ky| \to 0$, it increases logarithmically.

The asymptotic form written out above is inapplicable for small $|\sin\theta|$ and $|\cos\varphi|$. In this case, the stationary point $q = -\mathrm{ctg}\,\varphi$ turns out to be close to the branch points $q = \pm i\sin\theta$ and the following function, which does not reduce to well-known special functions, is the model integral

$$W(\alpha,\beta) = \int_{-\infty}^{\infty} \frac{\exp[i(x-\alpha)^2]dx}{\sqrt{\beta^2+x^2}} \tag{4.3}$$

The asymptotic form G for small $|\sin\theta|$ and $|\cos\varphi|$ is

$$G \approx -\frac{1}{2\pi^{3/2}V^2}\operatorname{Re}\left[\frac{\exp i(-\pi/4+kr\sin\varphi)}{\sqrt{kr(1+\sin\varphi\cos\theta)}}W\left(\sqrt{2kr}\cos\frac{\theta}{2}\sin\left(\frac{\pi}{4}-\frac{\varphi}{2}\right),\ \sqrt{2kr}\sin\frac{\theta}{2}\cos\left(\frac{\pi}{2}-\frac{\varphi}{2}\right)\right)\right]+$$

$$+\,G_2(r,\theta,\varphi) \tag{4.4}$$

The quantity $(kr)^{1/4}$ is the criterion of the smallness of $\sin\theta$ and $\cos\theta$ in the above estimates.

5. CONCLUSION

We will now formulate the results which have been obtained. A function G has been considered which has the integral representation (1.2). The field of the linear internal gravitational waves, excited in a horizontally stratified medium with a density distribution $\rho(z) = \rho_0\exp(-\sigma z)$ with a dipole mass source of unit moment, orientated along the x axis and which moves along the x axis in a negative direction at a velocity V, is expressed in terms of G using formulae (1.1). The asymptotic for of G when $\xi = x + Vt = r\cos\theta$; $y = r\sin\theta\cos\varphi$; $z = r\sin\theta\sin\varphi$, $k = \sqrt{(\sigma g)}/V$ and $kr \gg 1$ is described by the following expressions.

When $|z| > \sqrt{(r/k)}$, $|\cos\theta| > (kr)^{-1/4}$, that is, far from horizontal plane of the source $z = 0$ and the transverse plane $\theta = \pi/2$, it is described by formula (2.2).

Close to the transverse plane $\theta = \pi/2$, but beyond the horizon of the source, that is, when $|\cos\theta| < (kr)^{-1/4}$, $|z| > \sqrt{(r/k)}$, it is described by formula (2.3). Close to the horizontal plane of the source in the front half space, that is, when $|z| > \sqrt{(r/k)}$, $\pi/2 < \theta < \pi$, it is described by formula (3.5).

Close to the horizontal plane of the source in the rear half-space but far from the half-axis $\xi > 0$, that is, when $|z| > \sqrt{(r/k)}$, $0 < \theta < \pi/2$, $\sin\theta > (kr)^{-1/4}$, it is described by formula (4.1).

In the neighbourhood of the half-axis $\xi > 0$, that is, when $\theta \approx \sin\theta < (kr)^{-1/4}$, it is described by formula (4.2) when $\cos\varphi > (kr)^{-1/4}$ and by formula (4.3) when $\cos\varphi < (kr)^{-1/4}$.

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